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RESEARCH REPORT No. CX-40

Low Energy Scattering by a Compound System: Positrons on Hydrogen

LARRY SPRUCH and LEONARD ROSENBERG

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February 1959

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Abstract*

The formalism, developed by Kato, which gives upper and lower bounds on the phase shift for the scattering of a particle from a center of force is shown to have useful applications when the scatterer is a compound system. In particular, the problem of low energy positron scattering from atomic hydrogen, with zero total orbital angular momentum, is studied. It is shown that at zero energy the ordinary variational calculation, which ignores second order contributions, provides a bound on the scattering length, from which a bound on the cross section is deduced. For non-zero energies a bound on the phase shift may similarly be obtained, but for a fictitious problem with cut-off potentials. If the energy is sufficiently small (less than 3 e.v., say) the error thus incurred is expected to be negligibly small. Numerical calculations performed at $k = 0$ and $ka_o = 0.2$ lead to the result that the effects of polarization are large enough to cause the positron to be on the whole attracted to the hydrogen atom.

It is shown, independently of the Kato formalism, that in general a bound on the scattering length may be obtained from the ordinary variational calculation provided no composite bound state exists for the system.

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*A preliminary report was given at the Washington Meeting of the American Physical Society in April, 1958 [Bull. Am. Phys. Soc. Ser. II, 3, 171 (1958)]

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1. Introduction

It is somewhat disconcerting that more than thirty years after the advent of quantum theory there is in common use no general method that may be applied with confidence to the calculation of the cross section for scattering by a compound system. This is true even for compound systems consisting of only two particles. In the few cases in which the calculation is on firm theoretical grounds, such as the low energy scattering of neutrons from chemically bound protons[1], the reliability of the calculation depends on some quite special properties of the forces and of the scattering system.

It is of course true that there are a few scattering problems involving compound scatterers which do not possess any special properties for which the calculation can be accepted with a high degree of confidence. An example is the scattering of electrons by hydrogen atoms[2], [3] (for restricted ranges of scattering energies). The fact remains, nevertheless, that these calculations do not give any indication of their reliability; they do not provide any estimate of the error nor, contrary to the situation in the evaluation of the energy of the ground state, do they provide an upper or a lower bound on the cross section. Rather, our confidence is based on the fact that variational calculations performed with a variety of trial functions lead to results which do not differ appreciably among themselves, nor with a result based on the accurately determined bound state wave function of H^-[4]. The use of a variational method is in itself by no means sufficient to ensure any accuracy whatever. Since the effects of polarization can be profound, it may be quite difficult to determine a suitable form for the trial function, and of course even in a variational calculation a poor trial function generally gives a poor result. Further, the inclusion of more parameters in the trial function need not improve the result. In applications of the Kohn and Hulthén forms of the variational principle certain consistency criteria are frequently used to check the validity of the calculation[5]. It is quite clear, however, that these criteria are by no means completely satisfactory. The knowledge that the consistency ratio is very different from unity is, to be sure, a useful if negative piece of information; the difficulty is that a consistency ratio close to unity may imply nothing more than an accurate calculation within the limitations, however severe and inappropriate, of the assumed form of the wave function. Thus, Bransden, Dalgarno, John, and Seaton[3] offer the example of a trial function for the electron-hydrogen scattering problem in the form of a properly symmetrized product of one-particle functions. The assumed form gives rise to certain (approximate) one-body equations. If the one-particle functions are determined exactly as the solutions of these equations, perfect consistency will be achieved even if the resultant trial function bears no relationship to the true function. A still simpler example of the failure of the consistency criteria is found in the present e^+H problem. A trial function involving no polarization of the hydrogen atom will lead to perfect consistency if the positron wave function is the exact solution of the static problem, and yet polarization plays a crucial role. The weakness of the consistency checks is well recognized, but though all agree with a previous author[6] that "a foolish consistency is the hobgoblin of little minds", these checks have nevertheless been frequently applied simply because no better standards have been available.

In the case of scattering by a static central potential, where the cross section, σ, consists of a sum of partial cross sections, σ_L, each of which depends upon just one real number, the phase shift $\bar{\eta}_L$, a number of methods have been devised for determining upper and lower bounds on cot $\bar{\eta}_L$ and hence on σ_L[7]. By far the most general of these methods is the one due to Kato[8]. The method was applied by Kato to the case of $L=0$ with rather striking results. It has also been applied to higher angular momenta[9]; the calculations are then more cumbersome, but useful results can still be obtained. Some improvements in the formalism itself have also been effected, including the minimization of the amount of a priori knowledge of $\bar{\eta}_L$ which is required before the method can be applied[10]; this will generally be essential for the compound scatterer where (except for zero energy) one ordinarily has no a priori knowledge of $\bar{\eta}_L$ whatever.

It is to be noted that the Kato formalism is by no means restricted to quantum mechanical problems; it is applicable to a wide variety of problems, including, for example, the study of electromagnetic waveguides[11].

It is to be expected that some difficulties will arise in an attempt to generalize the Kato method to scattering by a compound system. These difficulties will be discussed in Section 2 where it will also be shown that there exist interesting circumstances under which they can be circumvented. It is further to be anticipated that a calculation which furnishes one or both bounds will be more complicated than a variational calculation. While this is generally the case, there are situations in which an ordinary variational calculation gives a bound.

Of the problems to which the Kato method can be applied we have thought it worthwhile to consider first the scattering of positrons by hydrogen atoms[12], [13]. This problem is of some interest itself; also, the methods involved can be extended in a natural way to the study of e^+ on He where a direct check between theory and experiment is possible[12], [14], [15]. The problem has the additional virtue of providing an opportunity for studying the effect of " specific polarization" , i.e., the distortion of the scattering system which is present in the wave function whether or not its antisymmetrization is required. Application of the Pauli principle automatically introduces some distortion, called " exchange polarization" , and in the case of electron scattering by hydrogen, for example, it is rather difficult to separate out these two "polarization" effects[16]. However, the primary reason for considering this problem first is its simplicity. To begin with, it is an atomic problem which, being free from the additional complexities of tensor forces and space exchange, presents a more convenient starting point for this type of investigation. The absence of the Pauli principle and the fact that no three body bound state exists further simplify the analysis. The "infinite" mass of the proton as compared to the electron or positron mass is also a simplification since the complexities that sometimes arise from the separation of the center of mass cannot occur. Of course, the possibility of positron-electron annihilation exists. If, however, we assume that the annihilation cross section, σ_a, is given by[14]

$$\sigma_a = (\text{very small constant})\, v^{-1} \qquad , \tag{1.1}$$

where v is the speed of the positron, it follows that only at extremely small energies will this process compete with elastic scattering. (Actually, the above formula ignores the velocity distribution of the electron in the atom, as well as the electrostatic interaction between positron and electron. These effects might well reduce the annihilation cross section.) Our subsequent " zero" energy results may therefore be interpreted as extrapolations from some small non-zero energy. The situation is quite similar to that which obtains in the case of " zero" energy neutron-proton scattering, where the competing capture process obeys a v^{-1} law and is significant only for the lowest energies. The only unpleasant feature associated with the $e^{+}H$ problem is the presence of the long range Coulomb potential. The numerical calculations are discussed in Section 4.

Some general remarks should be made about the utility of theoretical calculations of the type under consideration. We note, first, that to our knowledge no machine calculations have been performed which rigorously treat the scattering by a compound system. Further, in electron scattering by compound systems, the region of greatest interest is the low energy region which is at the present time relatively inaccessible to experimental investigation. It is true that one is primarily interested in electron scattering by oxygen and nitrogen, due to its importance in ionospheric processes, and the present methods cannot yet be applied to such a complicated case. On the other hand, a theoretical analysis of low energy singlet and triplet scattering of electrons by hydrogen atoms is possible, (work on this is in progress), and a close estimation of the cross section for this process should be useful in the determination of the accuracy of various approximations; those which are found to be reasonably accurate can then be made with a considerably greater degree of confidence in the more complicated problems. Finally, in the low energy nuclear scattering problems (neutrons on deuterons, for example), where the experiments can and have been performed, a theoretical analysis is still required for in these problems one is primarily interested in deducing the potential from the scattering data.

2. The Kato Method

A. The True Problem

To facilitate subsequent discussion we define a problem with somewhat specialized conditions. We consider the scattering of a particle (particle number 1) of mass m_1 from a system in its ground state consisting of two particles of masses M and m_2 respectively. We take $m_1 = m_2 = m$ and choose $M >> m$ so that the scattering system is effectively a particle (particle number 2) of mass m in a field of force. Particles 1 and 2 are assumed to be distinguishable so that we need not concern ourselves with the Pauli principle. Let E_{2g} denote the ground state energy of the second particle in the field of force; $\triangle E_2$ is its minimum excitation energy. Correspondingly, if the first particle can be bound to the center of force, let E_{1g} denote the ground state energy. If particles 1 and 2 can by themselves form one or more bound states we denote their ground state energy by E_{12g}. Finally, we let T_1 label the initial kinetic energy of the incident particle. The discussion is limited to scattering states of zero total orbital angular momentum.

The Kato method is applicable only if the scattering problem is such that the various angular momenta are not coupled and if the scattering for a given angular momentum can be completely characterized by one real phase shift. For the scattering of a particle by a center of force, both these conditions are realized if the potential is spherically symmetric. In the many body scattering problem we must not only have potentials which are independent of the spatial orientation of the system as a whole, to allow a partial wave analysis, but also T_1 must be restricted in magnitude if the characterization by one real phase shift is to be possible. In particular,

(1) Excitation or ionization must not be possible; i.e., we must have $T_1 < \Delta E_2$.

(2) Exchange must not be possible, requiring $|E_{2g}| > |E_{1g}|$, and further $T_1 < |E_{2g}| - |E_{1g}|$.

(3) Pick-up must not be possible, requiring $|E_{2g}| > |E_{12g}|$ and $T_1 < |E_{2g}| - |E_{12g}|$.

In the case of positron scattering from hydrogen atoms, we have $|E_{2g}| = \frac{1}{2} e^2/a_0$, $\Delta E_2 = \frac{3}{4} |E_{2g}|$, $E_{12g} = \frac{1}{2} E_{2g}$.

Exchange is not possible and conditions (1) and (3) above require that T_1 be less than $\frac{3}{4}|E_{2g}|$ and $\frac{1}{2}|E_{2g}|$, respectively. It is necessary therefore, that the initial positron energy be less than 6.8 e.v. in order that one real phase shift completely characterize the $L = 0$ scattering.

We return now to the more general problem and assume that conditions (1), (2) and (3) are satisfied. The differential equation to be solved is

$$(2.1)\quad (2m/\hbar^2)(E - H)\,\bar{u}_\theta(r_1,r_2,r_{12})/r_1 \equiv (1/r_1)\mathcal{L}\bar{u}_\theta(r_1,r_2,r_{12}) = 0 \quad ,$$

where H is the Hamiltonian of the system and $E = E_{2g} + T_1$ is the total energy. Here $r_1 = |\vec{r}_1|$, $r_2 = |\vec{r}_2|$, and $r_{12} = |\vec{r}_1 - \vec{r}_2|$, where $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the two particles measured from the center of force. Defining $\varepsilon_g = (2m/\hbar^2)|E_{2g}|$, $k^2 = (2m/\hbar^2)T_1$, and $W = -(2m/\hbar^2)V$ for all three "potentials" W_1, W_2 and W_{12}, we have

$$(2.2)\quad \mathcal{L} = \frac{\partial^2}{\partial r_1^2} + \frac{1}{r_2^2}\frac{\partial}{\partial r_2} r_2^2 \frac{\partial}{\partial r_2} + \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)\frac{\partial}{\partial p}(1 - p^2)\frac{\partial}{\partial p}$$

$$+ W_1(r_1) + W_2(r_2) + W_{12}(r_{12}) + k^2 - \varepsilon_g \quad ,$$

where p is the cosine of the angle between $\vec{r}_1$ and $\vec{r}_2$. It is sometimes more convenient to write the total kinetic energy operator, T, as

$$(2.3)\quad -2m/\hbar^2 T = \frac{1}{r_1^2}\frac{\partial}{\partial r_1} r_1^2 \frac{\partial}{\partial r_1} + \frac{1}{r_2^2}\frac{\partial}{\partial r_2} r_2^2 \frac{\partial}{\partial r_2}$$

$$+ 2\frac{1}{r_{12}^2}\frac{\partial}{\partial r_{12}} r_{12}^2 \frac{\partial}{\partial r_{12}} + \frac{(r_1^2 + r_{12}^2 - r_2^2)}{r_1 r_{12}} \frac{\partial^2}{\partial r_1 \partial r_{12}}$$

$$+ \frac{(r_2^2 + r_{12}^2 - r_1^2)}{r_2 r_{12}} \frac{\partial^2}{\partial r_2 \partial r_{12}} \quad .$$

The exact solution, $\bar{u}_\theta$, must satisfy the boundary conditions

$$\bar{u}_\theta \to \bar{R}(r_2)\left[\cos(kr_1+\theta)+\cot(\bar{\eta}-\theta)\sin(kr_1+\theta)\right], \quad \text{as } r_1 \to \infty$$

$$\text{(2.4)} \quad \bar{u}_\theta \to 0 , \qquad \text{as } r_2 \to \infty$$

$$\bar{u}_\theta = 0 \qquad \text{at } r_1 = 0$$

where $\bar{R}(r_2)$ is the exact ground state wave function for the bound particle, normalized such that

$$\text{(2.5)} \quad 2\int_c^\infty r_2{}^2 dr_2\, \bar{R}^2(r_2) = 1.$$

The normalization parameter, θ, satisfies $0 \leq \theta < \pi$ but is otherwise arbitrary. The true phase shift, $\bar{\eta}$, is determined by (2.1), (2.2) and (2.4). (We have here, and in the following, dropped the subscript zero on $\bar{\eta}$.)

We now introduce a trial function, $u_\theta(r_1, r_2, r_{12})$, which satisfies the same boundary conditions as $\bar{u}_\theta$ but with the true phase shift, $\bar{\eta}$, replaced by a trial phase shift, η. Note that this still involves the exact wave function, $\bar{R}(r_2)$. The identity

$$\text{(2.6)} \quad k\cot(\bar{\eta}-\theta) = k\cot(\eta-\theta) - \int u_\theta \mathcal{L} u_\theta d\tau + \int w_\theta \mathcal{L} w_\theta d\tau$$

then carries over from the one body case where now $d\tau = dr_1\, r_2{}^2\, dr_2\, dp$ and where

$$\text{(2.7)} \quad w_\theta(r_1,r_2,r_{12}) = u_\theta(r_1,r_2,r_{12}) - \bar{u}_\theta(r_1,r_2,r_{12}).$$

Since w_θ is a first order term, Eq. (2.6) constitutes a variational principle for cot $(\bar{\eta}-\theta)$ upon dropping the last term.

The requirement that the asymptotic form of u_θ be proportional to the exact wave function $\bar{R}(r_2)$ is evidently extremely restrictive. It is, however, necessary in order that the decomposition

$$(2.8) \qquad \int \bar{u}_\theta \mathcal{L} u_\theta \, d\tau = \mathcal{L} u_\theta \mathcal{L} u_\theta dr - \int w_\theta \mathcal{L} w_\theta d\tau$$

may be used. If the requirement is dropped the integrals on the right hand side of the above equation will not separately exist, each of them diverging for large r_1. (The *sum* of the two integrals is of course still finite.) The required knowledge of the ground state wave function formally eliminates the possibility of applying the Kato method to practically all scattering systems consisting of more than two particles. This difficulty, present in the ordinary variational problem as well, can fortunately be partially circumvented. Assuming a knowledge of the ground state wave function we introduce into the appropriate integral expressions a trial function which is the sum of an "inside wave function" [17] and a term which provides the correct asymptotic form; the operation of $\mathcal{L}$ on the asymptotic form is carried out formally. At *this* stage, we can replace the ground state wave function by some approximation to it without introducing any infinities. The error involved will be reproduced in our estimate of (or bound on) k cot $(\bar{\eta}-\theta)$ to *first* order, i.e., this error is not reduced by the fact that a variational formulation is being utilized. Further, the question as to whether the bound had been preserved would have to be examined. However, for the cases for which the ground state wave function is known with considerable precision, (He for example), the replacement should be irrelevant.

B. The Associated Eigenvalue Problem.

Consider the equation

$$\mathcal{L}\phi_\theta + \mu_\theta\, \rho\, \phi_\theta = 0 \tag{2.9}$$

where ρ is a non-negative function to be chosen such that the scattering problem determined by Eq. (2.9) and by appropriate boundary conditions is characterized for each value of μ_θ by one real phase shift, $\delta(\mu_\theta)$. Then there exists an infinite set of discrete eigenvalues $\mu_{n,\theta}$ and corresponding eigenfunctions $\phi_{n,\theta}$ normalized by

$$k^{-1}\int \phi_{n,\theta}\, \phi_{m,\theta}\, \rho\, d\tau = \delta_{nm} \quad , \qquad m,n = 0, \pm 1, \pm 2, \ldots \tag{2.10}$$

where δ_{nm} is the Kronecker δ-symbol, such that

$$\begin{aligned} &\phi_{n,\theta} \to \text{const. } \bar{R}(r_2) \sin(kr_1 + \theta + n\pi), &&\text{as } r_1 \to \infty \\ &\phi_{n,\theta} \to 0 \quad , &&\text{as } r_2 \to \infty \\ &\phi_{n,\theta} = 0 \quad , &&\text{for } r_1 = 0 . \end{aligned} \tag{2.11}$$

We denote the smallest positive eigenvalue of the set $\mu_{n,\theta}$ by α_θ and the smallest (in magnitude) negative eigenvalue by $-\beta_\theta$. The Kato inequalities are

$$-\alpha_\theta^{-1}\int (\mathcal{L}u_\theta)^2\, \rho^{-1} d\tau \le \int w_\theta \mathcal{L} w_\theta\, d\tau \le \beta_\theta^{-1}\int (\mathcal{L}u_\theta)^2\, \rho^{-1} d\tau \quad . \tag{2.12}$$

This formal development follows that for the one body problem, mutatis mutandis. (Some questions of completeness are discussed in Appendix B). However, the actual restrictions on ρ, referred to above, involve considerations peculiar to the many body problem. We observe first a restriction which does carry over from the one body case, namely, that ρ must vanish faster than $1/r_1$ as r_1 becomes infinite in order that normalization according to Eq. (2.10) be possible and the phase shift, $\delta(\mu)$, be defined. The choice $\rho = \rho(r_2)$, for example, is eliminated*. To eliminate pick-up in the associated eigenvalue problem, which is defined for an infinitely broad range of potential strengths, there are two alternatives.

a) Choose ρ to be independent of r_{12}, at least for r_1 and/or $r_2 \rightarrow \infty$. Then E_{12g} will be independent of μ.

b) Imagine the bound particle to be enclosed in a box with perfectly rigid walls. The size of the box, although finite, may be large enough so that the deviation from the real problem is negligibly small.

The exchange process may occur in the associated problem even if it is absent in the real problem. Again there are two alternative solutions.

a′) Choose ρ to be independent of r_1, at least for $r_2 \rightarrow \infty$. Then E_{1g} will be independent of μ.

b′) Put the bound particle in a box, as in b) above.

If this latter suggestion is adopted we require

$$(2.13) \qquad r_1\rho(r_1,r_2,r_{12}) \rightarrow 0 \quad , \qquad \text{as } r_1 \rightarrow \infty,$$

but ρ is otherwise arbitrary.

---

*The theorem which states that the phase shift for scattering by a center of force is a monotonic function of the potential strength for a potential of constant sign carries over to the many body case provided that the unperturbed energy of the compound system remains fixed as the potential strength varies. Thus, another reason for demanding that ρ vanish with large r_1 is that the required monotonicity of $\delta(\mu)$ with μ will not be preserved if ρ becomes some function of r_2 as $r_1 \rightarrow \infty$. Furthermore, for large negative values of μ, particle 2 would no longer be bound. The boundary conditions, Eq. (2.11), would then be violated.

3. Application to Positron Scattering From Hydrogen.

Kato has shown for the static central potential that if the potential cuts off outside a region of radius a then $\beta_\theta = \infty$ if, in addition, the following conditions are satisfied

(1) $ka < \pi - \theta$
(2) $\bar{\eta} < \theta < \pi$.

This is a particularly interesting case since then a lower bound may be obtained on the phase shift without the necessity of evaluating the more difficult integral, $\int (\mathcal{L}u_\theta)^2 \rho^{-1} d\tau$. The analogous result holds in the three body problem considered in Section 2. In this case the required proof that $\delta(-\infty) = -ka$ (where $\rho = 0$ for $r_1 > a$) is slightly more involved. We may write the solution for $r_1 > a$ as

$$(3.1) \qquad \phi = \text{const. } \bar{R}(r_2) \sin(kr_1 + \delta(\mu)) + \Big(\sum_{n,L} + \int \Big) \bar{R}_{n,L}(r_2)\, h_L^{(1)}(k_n r_1)\, P_L\,(p)$$

where $\bar{R}_{n,L}(r_2)$ are the excited inner particle bound states and the continuum solutions, $h_L^{(1)}$ are the spherical Hankel functions and P_L are the Legendre polynomials; k_n is defined by

$$(3.2) \qquad k^2 - \varepsilon_g = k_n^2 - \varepsilon_n \quad .$$

For $\mu = -\infty$ we must have $\phi(r_1 = a) = 0$. From the orthogonality of $\bar{R}$ with the functions $\bar{R}_{n,L}$ we have

$$(3.3) \qquad \sin(ka + \delta(-\infty)) = 0 \quad ,$$

from which the result follows.

The many body problem is generally complicated by the difficulty that one cannot get even crude bounds on $\bar{\eta}$, so that condition (2) cannot be proved, (assuming that it is true). We have recourse here to the use of a " conditional inequality" [10]. We simply *assume* that condition (2) is satisfied; the subsequent " conditional inequality" for $k \cot(\bar{\eta} - \theta)$ may or may not be valid. However, the bound on $\bar{\eta}$ deduced from the " conditional inequality" on $k \cot(\bar{\eta} - \theta)$ will be valid *whether or not* the bound on $k \cot(\bar{\eta} - \theta)$ is valid, and therefore *whether or not* condition (2) is in fact satisfied.

We therefore consider the following set of potentials

$$
\begin{aligned}
(3.4) \qquad W_2(r_2) &= 2(a_o r_2)^{-1} , &\qquad 0 \leq r_2 \leq b \\
&= c , &\qquad b < r_2 \leq b + d \\
&= -\infty , &\qquad r_2 > b + d
\end{aligned}
$$

$$
\begin{aligned}
(3.5) \qquad W_1(r_1) + W_{12}(r_{12}) &= -2(a_o r_1)^{-1} + 2(a_o r_{12})^{-1} , &\qquad r_1 \leq a \\
&= 0 &\qquad r_1 > a
\end{aligned}
$$

Here a_o is the Bohr radius. As *a* and *b* become infinite the above potentials approach those of the positron-hydrogen system. As indicated above, *b* may be chosen large enough so that the presence of the infinitely repulsive potential barrier has no effect on the real problem*, although it plays an important role in limiting the number of allowable processes which can take place in the associated problem. Furthermore, for *k*

---

*Strictly speaking, one should study the integral, $\int u_\theta \mathcal{L} u_\theta d\tau$, in the limit of large *b* , with the parameters *c* and *d* always adjusted to give the correct hydrogenic energy, ε_g; the error involved in changing the potential should then be shown to be arbitrarily small for b sufficiently large.

sufficiently small, a may be chosen large enough so that the neglect of the potential which may exist beyond $r_1 = a$ introduces a negligible error in the phase shift. If k is zero, we may let a become infinite, in which case we introduce no distortion of the positron-hydrogen potentials. In this case, the validity of the second condition stated above, namely $\bar{\eta} < \theta < \pi$, is deduced by assuming that no three-body bound state exists for the positron-hydrogen system (none has been found either experimentally or theoretically)[18]. We then apply a theorem which is the generalization to many body systems of a theorem proved by Levinson[19] for the scattering from a static potential. It states that when the exclusion principle is not in effect the phase shift for zero energy scattering is $n\pi$ where n is the number of bound states of the composite system[20]. From this we conclude that $\bar{\eta} = 0$ for $k = 0$. A rigorous proof of the above theorem has not been given even for potentials which fall off rapidly. A proof for the slowly decaying Coulomb forces will be even more difficult. We nevertheless apply the theorem with some confidence, for its generalization, which applies to electron scattering from atoms, has proved to be correct in all cases where it has been checked[21]. That the theorem should be applicable to the Coulomb case (assuming it applies to rapidly decaying potentials) is perhaps made more reasonable by the observation that the " effective interaction" of the positron or electron with the atomic system falls off fairly rapidly. For e^+ on H, e.g., it goes as $1/r^4$. Actually, as we will show in Section 5, the results we have obtained do not depend on the validity of the above mentioned theorem.

With the choice $\theta = \pi/2$ and $-\bar{A} \equiv \lim_{k \to 0} \bar{\eta}/k$, $-A \equiv \lim_{k \to 0} \eta/k$, $\hat{u} \equiv \lim_{k \to 0} u_{(1/2)\pi}/k$, we have the following bound on the scattering length, $\bar{A}$.

$$(3.6) \qquad \bar{A} \leq A - \int \hat{u}\mathcal{L}\hat{u}\, d\tau$$

where $\hat{u}$ has the asymptotic form

$$(3.7) \qquad \hat{u} \xrightarrow[r_1 \to \infty]{} \hat{u}_\infty \equiv (2/a_o{}^3)^{1/2}\, e^{-r_2/a_o}\,(A - r_1).$$

One of the strong features of the Kato method is that in general it provides a different bound for each choice of θ. This is not true at $k = 0$, however, where the identical result is obtained for any θ other than $\theta = 0$. The choice $\theta = 0$ will lead to a valid bound on $\bar{A}$ only if η approaches zero from below as k goes to zero, i.e., only if $\bar{A}$ is positive. In this case we have

$$(3.8) \qquad \bar{A}^{-1} \geq A^{-1} + A^{-2}\int \hat{u}\mathcal{L}\hat{u}\, d\tau \quad .$$

If $\hat{u}$ is sufficiently close to the true function such that A and $\bar{A}$ are of the same sign and such that $|\hat{I}| \ll |A|$, where

$$\hat{I} \equiv \int \hat{u}\mathcal{L}\hat{u}\, d\tau \quad ,$$

then Eq. (3.8) may be rewritten as

$$(3.8)' \qquad \bar{A} \leq A - \hat{I} + A^{-1}\hat{I}^2 + \ldots, \qquad\qquad |\hat{I}| \ll |A|.$$

It then follows (since $A > 0$) that Eq. (3.6) is superior to Eq. (3.8). Therefore, for the special case of zero energy scattering, where no composite bound state exists, the problem of the optimum choice of θ is particularly simple. One should use Eq. (3.6), which follows from any choice of θ satisfying $0 < \theta < \pi$.

4. Numerical Calculation

A. No Polarization Approximation, k = 0.

To begin with, we consider the class of trial functions of the form

$$(4.1)\qquad u(r_1,r_2,r_{12}) = g(r_2)f(r_1) \quad .$$

The required asymptotic form fixes $g(r_2)$ as

$$g(r_2) = \bar{R}(r_2) = (2/a_o{}^3)^{1/2}\, e^{-r_2/a_o}.$$

We also have

$$(4.2)\qquad f(o) = 0$$

$$f(r_1) \to A - r_1 \quad , \qquad\qquad \text{as } r_1 \to \infty.$$

A trial function in this product form corresponds to the approximation (variously referred to as the no polarization approximation, the static approximation, and the one body approximation) in which the hydrogen atom is not polarized by the incoming positron. We substitute the above expression for $\hat{u}$ into Eq. (3.6), and integrate over r_2 and **p**. Dropping the subscript on r_1 we find

$$(4.3)\qquad \bar{A} \leq A - \int_o^{\infty} f(r)\, \mathcal{L}_s\, f(r)dr$$

where

$$(4.4)\qquad \mathcal{L}_s = \frac{d^2}{dr^2} + W_s(r) \qquad ,$$

and

$$(4.5)\qquad W_s(r) = -(2/a_o)\, e^{-2r/a_o}\, (1/a_o + 1/r)$$

so that in the static approximation the **positron** is repelled by the hydrogen atom. The best possible choice for f(r) would be $\bar{f}_s(r)$, the exact wave function for scattering by a fixed potential, $W_s(r)$. This will now be shown to follow from an application of the Kato method to this one body problem*. Since no bound state exists for the potential $W_s(r) + \mu\rho(r)$ with μ negative we conclude that

$$(4.6) \qquad \delta(\mu) = 0 \qquad\qquad \mu < 0$$

and therefore that $\beta_{(1/2)\pi} = \infty$. Thus, for any trial function, f(r), we have

$$(4.7) \qquad \bar{A}_s \leq A - \int_0^\infty f(r) \mathcal{L}_s \, f(r) dr$$

where $\bar{A}_s$ is the scattering length for the static problem. Since the equality holds only when $f(r) = \bar{f}_s(r)$, the statement above has been verified. From Eq. (4.3) with $f(r) = \bar{f}_s(r)$ we find that

$$(4.8) \qquad \bar{A} \leq \bar{A}_s \quad .$$

In point of fact one cannot solve for $\bar{f}_s(r)$. However, since we are dealing here with the scattering by a static potential, at zero angular momentum and zero energy, it is a trivial matter to find an upper bound on $\bar{A}_s$ and hence on $\bar{A}$. We will also find a lower bound on $\bar{A}_s$ in order to measure the accuracy of our determination of $\bar{A}_s$ and to exhibit once again the power of the Kato method for this type of problem.

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*It should be emphasized that in the remainder of sub-section A various quantities, and in particular, β_θ and α_θ, refer to this (static) one body problem and not to the true problem.

We have already shown that $\beta_{(1/2)\pi} = \infty$. Choosing $\rho(r) = -W_s(r)$ we observe that $\alpha_{(1/2)\pi}$ is the smallest positive value of μ for which the " potential" $W_s(r)(1-\mu)$ can support a bound state, for the appearance of a bound state indicates that the phase shift has jumped from zero to π. (Actually, $\delta(\alpha_{(1/2)\pi}) = (1/2)\pi$.) Since $W_s(r)(1-\mu)$ isn't even attractive until $\mu = 1$, we immediately obtain $\alpha_{(1/2)\pi} > 1$. This lower bound can be improved quite easily. Bargmann[22] has shown that a necessary condition for the existence of a bound state for a potential $U(r)$ is

$$(4.9) \qquad \int_0^\infty r\,|U(r)|\,dr \geq 1.$$

With $U(r) = -(1-\mu)W_s(r)$, this leads to $\alpha_{(1/2)\pi} > 5/3$.*

If we choose

$$(4.10) \qquad f(r) = A(1-e^{-nr/a_o}) - r(1+Be^{-qr/a_o})$$

we find, with $n = 1.5$ and $q = 2$, that

$$(4.11) \qquad 0.5762a_o \leq \bar{A}_s \leq 0.5823a_o$$

where $A = 0.5904a_o$, $B = -0.1819$ for the upper bound calculation and $A = 0.5816a_o$, $B = -0.0698$ for the lower bound. (Even with $B = 0$ the bounds are within 5 °/o of each other.) At the same time we have

$$(4.12) \qquad \bar{A} \leq 0.5823a_o \quad .$$

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*A comparison of $(1-\mu)W_s(r)$ with the Hulthén potential[23] leads to $\alpha_{(1/2)\pi} > 2$. Although we shall use this fact later in a slightly different connection, we prefer to base the present result on an application of the Bargmann theorem rather than on this rare occurrence of an excellent comparison potential for which solutions may be obtained analytically.

The cross section at zero energy, $\sigma(k = o)$, is $4\pi\bar{A}^2$. The above result without an accompanying lower bound does not exclude a large negative value of $\bar{A}$ and therefore does not provide an upper or lower bound on $\sigma(k = o)$.

It is interesting to note that a lower bound on $\bar{A}_s$ could have been determined without performing a variational calculation. (Since this is a lower bound on $\bar{A}_s$, it does not provide any bound on $\bar{A}$.) It has been shown[10] that for two "potentials" $+W(r)$ and $-W(r)$ with phase shifts $\bar{\eta}_+$ and $\bar{\eta}_-$, respectively, where $W(r) > 0$, $\bar{\eta}_+ < \pi$ and $\bar{\eta}_- > -\pi$, the following inequality holds:

$$(4.13) \qquad k \cot\bar{\eta}_+ - k \cot\bar{\eta}_- \leq 2\, k \cot\eta_{+B} \quad .$$

Here η_{+B} is the value of $\bar{\eta}_+$ calculated in the Born approximation. For $W(r)$ we choose

$$(4.14) \qquad W(r) = -W_s(r) = (2/a_o)\, e^{-2r/a_o}\, (1/a_o + 1/r) \quad ,$$

the static "potential" for electron scattering by hydrogen. Since $-W(r)$ obviously cannot support a bound state, and since it can be shown* that $W(r)$ cannot support a bound state either, we have $\bar{\eta}_{+s} = \bar{\eta}_{-s} = 0$, for $k = 0$. Bounds on the $L = 0$ phase shift, $\bar{\eta}_{+s}$, for the above potential have been determined. In the limit of zero energy we have[10]

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*Here we do take recourse in a comparison with the Hulthen potential.

$$(4.15) \qquad 0.10560\ a_o^{-1} \leq (k \cot \bar{\eta}_{+s})_{k=o} \leq 0.10592\ a_o^{-1} \quad .$$

Now, as k -> o, the Born approximation yields

$$(4.16) \qquad k \cot \eta_{+sB} \to -1/\int_o^\infty r^2\, W_s(r)dr = 1/a_o \quad .$$

It therefore follows that

$$(4.17) \qquad (k \cot \bar{\eta}_{-s})_{k=o} \geq -2a_o^{-1} + 0.10560\ a_o^{-1}$$

and

$$(4.18) \qquad \bar{A}_s = -(k \cot \bar{\eta}_{-s})_{k=o}^{-1} \geq 0.5273\ a_o \quad .$$

While this result is not as good as the one given previously, it was obtained without constructing a trial function.

Summarizing the results of this sub-section, we find that an analysis of the static problem leads to an upper bound on the true scattering length, $\bar{A}$, but does not provide any information on a lower bound for $\bar{A}$. Further, the least upper bound obtainable in the static approximation, namely that given by Eq. (4.3), is positive so that we do not obtain an upper or a lower bound on $\sigma(k=o)$. Actually, as we will show in the next sub-section, $\bar{A}$ is negative so that the static approximation is totally inadequate in this problem.

B. Polarization Considered, k = o.

A trial function of the form $\hat{u} = g(r_1, r_2)$ may take into account, to some extent, the polarization of the hydrogen atom. As we can see, using arguments quite similar to those employed in sub-section 4A, this type of trial function leads to a positive scattering length. Thus, substituting $g(r_1,r_2)$ into Eq. (3.6) and integrating over p we find

$$(4.19)\quad \bar{A} \leq A - 2\int_0^\infty dr_1 \int_0^\infty r_2^{\,2}\, dr_2\, g(r_1,r_2)\,\mathcal{L}_t\, g(r_1,r_2)$$

where

$$(4.20)\quad \mathcal{L}_t = \frac{\partial^2}{\partial r_1^{\,2}} + \frac{1}{r_2^{\,2}}\frac{\partial}{\partial r_2} r_2^{\,2}\frac{\partial}{\partial r_2} + \frac{2}{a_o}\left(\frac{1}{r_2} + \frac{1}{r_>} - \frac{1}{r_1}\right) - \varepsilon_g \quad .$$

Here $r_>$ is the larger of r_1 and r_2. Since no bound state exists for the operator $\mathcal{L}$, none exists for $\mathcal{L}_t$. This is a consequence of the fact that the ground state energy satisfies a minimum principle and the space on which $\mathcal{L}_t$ is defined is a sub-space of that for $\mathcal{L}$. (The same result also follows directly from the fact that $1/r_> - 1/r_1$ is always negative[24].) Therefore, the exact scattering length, A_t, for $\mathcal{L}_t$ is bounded by the Kato inequality with $\beta_{(1/2)\pi} = \infty$. It then follows that $\bar{A}_t$ represents the best (i.e., minimum) result obtainable with a trial function which is independent of r_{12}. Since the "positron" is always repelled (i.e., $1/r_> - 1/r_< \leq 0$) the monotonicity theorem* shows that $\bar{A}_t$ is itself positive, which verifies our original statement.

It is clear that a realistic trial function should depend on r_1, r_2

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*See footnote on page 12.

and r_{12}. This dependence was chosen under the physically plausible assumption that the polarization is largely accounted for by the virtual formation of positronium. Since the latter has a ground state function proportional to $\exp(-1/2\, r_{12}/a_o)$ the term $C \exp(-tr_{12}/a_o - vr_2/a_o)$ - the additional factor being required for convergence - was added to a function similar in form to the trial function, Eq. (4.10), which was so successful in the static calculation. We have chosen

$$(4.21) \quad \hat{u}(r_1, r_2, r_{12}) = (2/a_o^3)^{1/2} \Big[A e^{-r_2/a_o} (1 - e^{-r_1/a_o}) - r_1 e^{-r_2/a_o} + Br_1\, e^{-qr_1/a_o - sr_2/a_o} + Cr_1\, e^{-tr_{12}/a_o - vr_2/a_o} \Big]$$

The presence of excited states of positronium would spread out the wave function and one would therefore expect that a value of t slightly less than $\frac{1}{2}$ would be the best choice. A number of sets of values of the exponential parameters q, s, t, and v were tried, but nothing even approaching a systematic determination of the best set was attempted. "Best" is here defined as the set which gives the lowest upper bound on $\bar{A}$. Since the results were found to be quite sensitive to the choice of the exponential parameters the realization that $\bar{A}$ satisfies a minimum principle was a great computational advantage. The linear parameters were of course determined variationally for each choice of the exponential parameters. Results are given in Table I.

The effect of polarization is rather dramatic. The value of $\bar{A}$ is negative so that, contrary to the situation in the static approximation, the positron is on the whole attracted to the hydrogen atom. Further, an upper bound on $\bar{A}$ now yields a lower bound on $\sigma(k=o)$.

Table I.

Results for $k = o$, with the trial function given by Eqs. (4.21) and (4.23), corresponding to the set of exponential parameters $\{q,s,t,v\} = \{0.50, 0.95, 0.46, 0.78\}$. The consistency ratio, R_k, is defined as $R_k = (A - \int \hat{u}\mathcal{L}\hat{u}\, d\tau)/A$. The second calculation yielded a trial function containing no nodes. Note that while the second result is an improvement over the first, its consistency ratio deviates further from unity.

A/a_o	B	C	D	upper bound on $\bar{A}/a_o$	lower bound on $\sigma(k=o)$	R_k
-1.3714	2.6516	-1.4620	0	-1.356	$7.357\pi a_o^2$	0.989
-1.4453	2.8106	-1.4979	-0.0532	-1.397	$7.802\pi a_o^2$	0.966

We note incidentally that the best value of t among the sets considered was $t = 0.46$, i.e., somewhat smaller than $\frac{1}{2}$, as expected.

Massey and Moussa[12] used a trial function of the form

$$(4.22)\quad u(r_1,r_2,r_{12}) = (2/a_o^3)^{1/2}\, e^{-r_2/a_o} \Big[\sin kr_1 + (a + (b+cr_{12})\, e^{-r_1/a_o})\,(1 - e^{-r_1/a_o}) \cos kr_1 \Big] \quad .$$

They determined $\bar{\eta}$ variationally for various values of k, the smallest of which was $k = 0.2/a_o$. Presumably they avoided $k = o$ because they did not trust the form of their trial function for that value of k. If, nevertheless, we use their form we find $\bar{A} \approx 0.512\ a_o$; this value does not differ appreciably from that obtained under the static approximation. It should be noted that despite the extremely poor value for $\bar{A}$ the above calculation provides a consistency ratio

$$R_K = (A - \int \hat{u} \mathcal{L} \hat{u}\, d\tau)\ /A \ = \ 1.04 \quad .$$

It is rather surprising that the two trial functions, Eqs. (4.21) and (4.22), each containing r_{12} dependence and three variational parameters, lead to such strikingly different results, corresponding as they do to an effective attraction in one case and to an effective repulsion in the other. These results speak strongly for the need for obtaining bounds.

It may well be that the possibility of the virtual formation of positronium makes the present problem particularly sensitive to the choice of the trial function, much more so than in the case of electron scattering by hydrogen, for example.

It would be difficult to say how reliable our result is without obtaining a lower bound on $\bar{A}$. We have been unable to do so, for we have been unable to obtain a lower bound on α_θ. (The difficulty is connected with our inability to prove formally that a composite bound state for $e^+ + H$ does not exist.) We therefore sought additional criteria for the accuracy of our results.

In particular, since no composite bound state exists, we expect by analogy with the one body problem that $\hat{u}$ should be nodeless. The trial function given by Eq. (4.21) has nodes, although at physically uninteresting points (r_2 large). Since attempts to remedy this by varying the exponential parameters were unsuccessful another term was added. This term was taken to be proportional to $\exp(-\delta r_{12}/a_o)$; if one thinks of this term as being related to the virtual formation of positronium in its first excited state (it does not have precisely the correct form), one might expect that δ should be somewhere near 1/4. Further, since our previous exponential factors directly coupled r_2 and r_1, and r_2 and r_{12}, we have in this case chosen the additional term to be proportional to $\exp(-2r_1/a_o)$. The final form is

$$(4.23)\quad \hat{u}(r_1,r_2,r_{12}) = u\ (\text{Eq.}(4.21)) + (2/a_o{}^3)^{1/2}\, Dr_1 e^{-r_{12}/4a_o - 2r_1/a_o}$$

The results are collected in Table I.

The linear parameters were determined variationally, i.e., without regard to the question of nodes; the resultant trial function is nevertheless nodeless. Further, the bound on $\bar{A}$ was lowered only slightly.

These two results suggest (in the absence of the other bound on $\bar{A}$ we do not use a stronger word) that our bound on σ may well be reasonably close to the true value. On the other hand our trial function does not indicate the presence of an effective polarization potential which falls off asymptotically like $1/r^4$, as demanded by perturbation theory[3].

Our trial function, Eq. (4.23), may now be used to estimate the effective range, r_o, for the e^+H system, where r_o is defined by

$$(4.24) \qquad r_o = 2\int (\hat{u}_\infty{}^2 - u^2)\, d\tau \qquad .$$

The asymptotic function, u_∞, is defined by Eq. (3.7). We find

$$(4.25) \qquad r_o \approx 11a_o \qquad .$$

This very large value of the effective range severely restricts the domain of applicability of the shape independent approximation,

$$(4.26) \qquad k \cot \bar{\eta} \approx -\bar{A}^{-1} + \tfrac{1}{2}k^2 r_o \; .$$

At the very best (if the coefficient of $k^4 r_o{}^3$ in the next term is small[25] and if the expansion is in fact of the same form as for the one body case*), Eq. (4.26) should be useful only for kr_o somewhat less than unity, i.e., ka_o somewhat less than 0.1. We shall return to this point later.

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*In the e^+H or e^-H problem, for example, $\bar{\eta}_L$ does not vanish with k like k^{2L+1} for $L>0$(due to the long range polarization potential[3]). Here the effective range theory, in its usual form, would not be applicable.

It might well have been less time-consuming, for all of our calculations, to have fixed the exponential parameters arbitrarily and to have used more terms containing linear parameters. The present approach was used in order to have a good trial function in reasonably simple form for other applications; in particular, we will now consider a calculation at a non-vanishing energy with a trial function which is a direct extention of the zero energy form given in Eq. (4.23).

C. Polarization Considered, $k \neq o$

In order to show that $\beta_\theta = \infty$ without introducing cut-off potentials we have restricted ourselves to the special case of zero energy scattering. It is not to be thought, however, that calculations leading to a bound on the cross section can be performed only at zero energy. As indicated in Section 3, for sufficiently small non-zero energies the cut-off point may be made large enough so that the difference between $\bar{\eta}$ and $\bar{\eta}_c$ (the phase shift in the presence of the cut-off potentials) is negligibly small. To illustrate the procedure we describe here a determination of a lower bound on $\bar{\eta}_c$ for $ka_o = 0.2$.

To begin with we ignore the question of obtaining a bound and perform an ordinary variational calculation for the true problem. We use a trial function which reduces, as $k \to 0$, to the zero energy function, Eq. (4.23), namely

$$(4.27)\quad u_{(1/2)\pi}(r_1,r_2,r_{12})/k = (2/a_o{}^3)^{1/2}\Big[Ae^{-r_2/a_o}(1-e^{-r_1/a_o})\cos kr_1 - e^{-r_2/a_o}\sin kr_1/k + Br_1e^{-qr_1/a_o - sr_2/a_o} + Cr_1e^{-tr_{12}/a_o - vr_2/a_o} + Dr_1e^{-r_{12}/4a_o - 2r_1/a_o}\Big] .$$

From the asymptotic form of this function we see that $A = -\tan\eta/k$. The variational principle is now

$$(4.28)\quad k\tan\bar{\eta} \approx k\tan\eta + \int u_{(1/2)\pi}\,\mathcal{L}\,u_{(1/2)\pi}\,d\tau$$

where $\mathcal{L}$ is given by Eq. (2.2). The result of the calculation is

$$\eta = 0.1645\ ,$$

$$I = \int u_{(1/2)\pi}\mathcal{L}u_{(1/2)\pi}\,d\tau = -0.0017/a_0\ ,$$

and $\bar{\eta} \approx 0.156$. As a point of comparison we note that a variational calculation (including polarization) performed by Massey and Moussa[12] for the same scattering energy gave $\bar{\eta} \approx -0.098$.

Since the trial function is based on the zero energy function which we expect to be quite good, we have obtained what we believe to be an accurate estimate of $\bar{\eta}$, although not necessarily a lower bound. Before considering the calculation involving the cut-off potentials we recall that the cut-off point, a, is limited by the condition $ka < \pi - \theta$. In order to be able to pick the largest value of a (for fixed k) we wish to choose the smallest value of θ consistent with the condition $\bar{\eta} < \theta$. (Note that we desire that this relation be satisfied, in order that our bound be a useful one. As discussed previously, the bound will be valid in either case.) The trial function, Eq. (4.27), is normalized with $\theta = (1/2)\pi$. This would lead to $a = 7.9a_0$. In order to obtain a variational estimate of $\bar{\eta}$ corresponding to the use of a trial function with different (and smaller) θ - normalization, we may simply choose

$$(4.29)\quad u_\theta = u_{(1/2)\pi}\cos\eta/\sin(\eta-\theta)\ ,$$

so that

$$(4.30) \qquad k \cot(\bar{\eta} - \theta) \approx k \cot(\eta - \theta) - \left[\cos \eta / \sin(\eta - \theta)\right]^2 I$$

where η and I have already been determined. We expect that this variational estimate of $\bar{\eta}$ should change very slowly as θ is decreased from $(1/2)\pi$. It is found, in fact, that to three significant figures the result is unchanged for θ as small as 0.34. (Smaller values of θ lead to appreciably lower estimates of $\bar{\eta}$.) We have therefore chosen $\theta = 0.3400$ which allows us to pick $a = 14a_o$.

To obtain a bound on $\bar{\eta}_c$ we must now perform a similar calculation with $\mathcal{L}$ replaced by $\mathcal{L}_c$ which contains the cut-off potentials, Eq. (3.5). The trial function, $u_{\theta,c}$, must now satisfy the additional requirement that $\mathcal{L}_c u_{\theta,c} = o$ for $r_1 \geq a$*, while $u_{\theta,c}$ must of course be continuous in slope and value. Such a function may be written as

$$(4.31) \quad u_{\theta,c}\left(\cos \eta / \sin(\eta - \theta)\right)^{-1} = u_{(1/2)\pi}\ (\text{Eq. } (4.27))\ g(r_1) + (2/a_o{}^3)^{1/2}(1 - g(r_1))\, e^{-r_2/a_o} \sin(kr_1 + \eta)/\cos \eta$$

where

$$(4.32) \qquad \begin{aligned} g(o) &= 1 \\ g'(a) &= o \\ g(r_1) &= o \quad , \qquad r_1 \geq a \quad . \end{aligned}$$

If $g(r_1)$ is close to unity throughout the region $r_1 < a$ we might expect that the results obtained will not differ very much from those of the above variational calculation. With this in mind we chose

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*This may be seen from the fact that in order to obtain $\beta_\theta = \infty$, ρ must vanish beyond $r_1 = a$. However, $\int (\mathcal{L} u_{\theta,c})^2 \rho^{-1} d\tau$ should exist.

$$(4.33) \qquad g(r_1) = \left[\frac{1 - e^{-\zeta(1 - r_1/a)}}{1 - e^{-\zeta}}\right]^2 \qquad r_1 \le a$$

$$= 0 \qquad r_1 > a$$

where ζ is arbitrary. If we let ζ take on a large, positive value ($\zeta \approx 5a$, say), $g(r_1)$ will have the desired shape.

We may write

$$(4.34) \qquad \int u_{\theta,c}\,\mathcal{L}_c\,u_{\theta,c}\,d\tau = \int u_\theta\,\mathcal{L}\,u_\theta\,d\tau + \mathcal{R} \;.$$

Due to the shape of $g(r_1)$, contributions to the integral $\mathcal{R}$ are very small for $r_1 \lesssim a$; for $r_1 \gtrsim a$ the expression is dominated by exponential factors which, owing to the large value of a, are also very small. It is in fact easily determined that to the accuracy of our calculation $\mathcal{R}$ is small enough to be neglected entirely. Therefore, the variational estimate of $\bar{\eta}$ is actually a bound on $\bar{\eta}_c$ and hence is *extremely* likely to be a bound on $\bar{\eta}$ itself, since the contribution of the potential in the region $r_1 > a$ is expected to be extremely small. Results appear in Table II.

It should be mentioned that even neglecting the difference between $\bar{\eta}$ and $\bar{\eta}_c$ we still do not, contrary to the zero energy case, obtain a rigorous bound on the cross section, defined by

$$(4.35) \qquad \sigma(k) = (4\pi/k^2)\,\sin^2\bar{\eta} \quad .$$

The substitution of $\bar{\eta}_{\text{lower}}$, the calculated lower bound on the phase shift, into Eq. (4.35) will yield a lower bound on $\sigma(k)$ *provided*

Table II

Results for $ka_o = 0.2$, with the trial function given by Eqs. (4.27), (4.31) and (4.33). The exponential parameters in Eq. (4.27) are the same as those cited in Table I. The bound is obtained on $\bar{\eta}_c$, the phase shift for the problem in which the positron interaction vanishes outside a sphere of radius $14a_o$. A variational calculation by Massey and Moussa,[11], for the same energy, gave $\bar{\eta} \approx -0.098$.

A/a_o	B	C	D	Lower bound on $\bar{\eta}_c$
- 0.8285	2.0434	- 1.1271	0	0.150
- 0.8300	2.1159	- 1.1464	-0.0412	0.156

$\bar{\eta} < \pi - \bar{\eta}_{lower}$. Such an upper bound has not been obtained; however, in view of the expected accuracy of our variational calculation, it is almost surely valid. The difficulty does not arise in the limiting case of zero energy since there the corresponding piece of information is supplied by the knowledge that no composite e^+H bound state exists.

It is easily shown, in the one body problem, that if the potential vanishes beyond $r = a$, and further, if $ka + \bar{\eta} < \pi$, then the wave function is nodeless in the region $r < a$. Thus, the standard definition of the phase shift, i.e.,

$$(4.36) \qquad \bar{\eta} = \lim_{n \to \infty} \left(n\pi - k\rho_n \right)$$

can be written, in this case, as

$$(4.36') \qquad \bar{\eta} = n\pi - k\rho_n \quad , \qquad \rho_n \geq a \quad ,$$

where ρ_n is the nth zero (not including the zero at the origin) in the wave function. Now consider the point $r' = (\pi - \bar{\eta})/k$. From the assumed restriction on $\bar{\eta}$ it is clear that $r' > a$. Now the wave function takes on its asymptotic form beyond the point *a*, and must therefore vanish at r'. But from Eq. (4.36)$'$, $r' = \rho_1$, so that we have shown that the first zero occurs beyond *a*. In the present problem, under similar conditions, (*i.e.*, $W_1 + W_{12} = 0$ for $r_1 > a$, and $ka + \bar{\eta} < \pi$) it may be shown that the function

$$(4.37) \qquad g(r_1) \equiv \iint r_2^2 \, dr_2 \, dp \, \bar{R}(r_2) \, \bar{u}(r_1, r_2, p)$$

is nodeless for $r_1 < a$. For *a* large enough it seems reasonable (we have not proved it) that a stronger statement can be made, namely, that $\bar{u}$ itself is nodeless for $r_1 < a$. If we assume this statement to be true, it may be used to test our trial function. With $D = o$ the function has nodes for $r_1 < a$ but these nodes do in fact disappear when the full trial function is used. Further, this improvement in the trial function is accompanied by only a small improvement in the bound on the phase shift. Both of these considerations suggest that this bound may well be in the neighborhood of the true value.

We have previously noted that the shape independent approximation can at best be expected to be applicable for $ka_o \leq 0.1$. If, nevertheless, we apply Eq. (4.26) to the present case, $ka_o = 0.2$, we find $\tan \bar{\eta} \approx 0.21$. This differs considerably from the variational result, $\tan \bar{\eta} \approx 0.16$. The difference is very likely due to the inapplicability of Eq. (4.26), but we may not exclude the possibility that our calculated value of r_o is too large due to inaccuracies in our zero energy wave function. It is of course also possible, though less likely, that our variationally calculated values of $\tan \bar{\eta}/k$ for $k = o$ and 0.2 are not close enough to the correct values.

6. Discussion

We have shown, by providing a particular example, that it is not only possible to obtain a bound on the cross section for scattering by a compound system but, under certain circumstances, that no additional work beyond that of performing a variational calculation is required. This is true in the case of zero energy scattering when no composite bound state exists. For slightly larger energies the bound obtained is no longer rigorous, since the interaction between the scattered particle and the scattering system is neglected outside a certain region. We believe, however, that in the example presented above, where the cut-off point is at $14a_0$, the neglected potential contributes practically nothing to the phase shift and that our bound is extremely likely to be valid.

The fact that the variational principle for cot $\bar{\eta}$ may, in special cases, be a minimum principle as well has some interesting consequences. Generalizing the results obtained previously, we may state that the scattering length for positron scattering from a static atom is greater than the true scattering length, $\bar{A}$, provided that the positron and the atom do not form a bound state. Also, a variational estimate of $\bar{A}$ will necessarily be positive in this case unless the trial function contains some dependence on the separation between the positron and one or more of the electrons. (This is analogous to a result obtained by Ore[24] in his investigation of the stability of simple positron compounds.) Another result of this formalism which would be otherwise rather difficult to prove is that under the conditions for which $\beta_{(1/2)\pi} = \infty$ the Kohn form of the variational principle for the phase shift (derivable from the Kato form by setting $\Theta = \pi/2$) has at least one important advantage over the Hulthén formulation. That is, an increase in the number of parameters in the trial function guarantees an improved result. This remark applies, for example, to the problem of zero energy triplet scattering of electrons from hydrogen.

The simple result that at zero energy the ordinary variational calculation provides a bound if the system has no composite bound state suggests that the Kato mechanism is not necessary in this case, since the above conditions make no reference to the associated eigenvalue problem. This is easily verified by observing that in the absence of a bound state the operator $\mathcal{L}$ is negative definite in the space of quadrically integrable functions. Now for $0 < \theta < \pi$,

$$(5.1) \qquad \hat{w}_\theta \equiv \lim_{k \to o} w_\theta / k$$

has the form of a zero energy bound state wave function. That is

$$(5.2) \qquad \hat{w}_\theta \to \text{const. } \bar{R}\,(r_2) \quad , \qquad \text{for } r_1 \to \infty.$$

Thus w_θ may be considered to be the limit of a sequence of functions, $w_\theta(\varepsilon)$, which have the asymptotic form

$$(5.3) \qquad \hat{w}_\theta(\varepsilon) \to \text{const. } \bar{R}\,(r_2)\, e^{-\varepsilon r_1} , \qquad \text{for } r_1 \to \infty .$$

Since

$$(5.4) \qquad \int \hat{w}_\theta(\varepsilon)\, \mathcal{L}\, w_\theta(\varepsilon) d\tau \quad < \quad 0$$

we conclude that this is true for $\varepsilon = o$ as well. This, along with Eq. (2.6), gives us the desired result. Note that

$$w_0 \to \text{const. } \bar{R}(r_2) r_1 \quad , \qquad \text{for large } r_1,$$

so that the theorem is not true in general for this choice of θ - normalization. We had also come to this conclusion from the point of view of the Kato formalism. (See Section 3).

While this proof merely reproduces a result previously obtained it is still quite interesting. First, our conclusions are freed from any dependence on the validity of the generalization of Levinson's theorem (see Section 3). More significantly, the type of reasoning involved points directly to the extension which includes situations in which bound states do exist. Here the results we have obtained do not follow, in any simple way at least, from the Kato formalism and may provide a considerable improvement. Furthermore, we have been led to the development of a method, valid at higher energies, which reproduces all results of Kato for which $\beta_\theta = \infty$ and which should otherwise provide a useful supplement to the Kato method. This work will be reported on in the future.

Appendix A

In the numerical evaluation of $\int \hat{u} \mathcal{L} \hat{u} \, d\tau$ (Section 4B) various integrals of the form

$$(A.1) \qquad I(\alpha,\beta,\gamma;\ell,m,n) \equiv \int_0^\infty dr_1 \int_0^\infty dr_2 \int_{|r_1-r_2|}^{r_1+r_2} dr_{12} e^{-\alpha r_2-\beta r_1-\gamma r_{12}} r_2^\ell r_1^m r_{12}^n$$

occur. These integrals may be generated from the basic integral

$$(A.2) \qquad I(\alpha,\beta,\gamma;0,0,0) = \frac{2}{(\alpha+\beta)(\beta+\gamma)(\alpha+\gamma)} \equiv \frac{2}{\delta}$$

using symmetry relations such as

$$(A.3) \qquad I(\alpha,\beta,\gamma;\ell,m,n) = I(\beta,\alpha,\gamma;m,\ell,n)$$

and recursion relations such as

$$(A.4) \qquad I(\alpha,\beta,\gamma;\ell+1,m,n) = -\frac{\partial}{\partial\alpha} I(\alpha,\beta,\gamma;\ell,m,n)$$

and

$$(A.5) \qquad I(\alpha,\beta,\gamma;\ell-1,m,n) = \int_\alpha^\infty I(\alpha',\beta,\gamma;\ell,m,n)\, d\alpha' .$$

We list below expressions for those integrals, involving non-vanishing values of γ, which entered into the calculation.

$$\begin{aligned}
I(\alpha,\beta,\gamma;\,1,0,0) &= \frac{2}{\delta}\left[\frac{1}{\alpha+\beta}+\frac{1}{\alpha+\gamma}\right]\\
I(\alpha,\beta,\gamma;\,1,1,0) &= \frac{4}{\delta^2}\left[\frac{\delta}{(\alpha+\beta)^2}+\alpha+\beta+\gamma\right]\\
I(\alpha,\beta,\gamma;\,1,0,1) &= \frac{4}{\delta^2}\left[\frac{\delta}{\alpha+\gamma}+\alpha+\beta+\gamma\right]\\
I(\alpha,\beta,\gamma;\,0,1,1) &= \frac{4}{\delta^2}\left[\frac{\delta}{\beta+\gamma}+\alpha+\beta+\gamma\right]\\
I(\alpha,\beta,\gamma;\,1,1,1) &= \frac{8}{\delta^2}\left[2+\frac{\alpha}{\beta+\gamma}+\frac{\beta}{\gamma+\alpha}+\frac{\gamma}{\alpha+\beta}\right]\\
I(\alpha,\beta,\gamma;\,1,-1,1) &= -\frac{16\alpha\gamma}{(\alpha^2-\gamma^2)^3}\log\frac{\beta+\alpha}{\beta+\gamma}+\frac{4}{(\alpha^2-\gamma^2)^2}\left[\frac{\gamma}{\beta+\alpha}+\frac{\alpha}{\beta+\gamma}\right].
\end{aligned}\tag{A.6}$$

In the calculation involving the non-vanishing scattering energy (Section 4C) additional integrals had to be evaluated. The results may be stated in terms of the general form

$$I(\alpha,\beta,\gamma;\,\ell,m,n;\,p,q) \equiv \tag{A.7}$$

$$\int_0^\infty dr_1 \int_0^\infty dr_2 \int_{|r_1-r_2|}^{r_1+r_2} dr_{12}\; e^{-\alpha r_2-\beta r_1-\gamma r_{12}}\, r_2^{\ell}\, r_1^{m}\, r_{12}^{n} \cos^p kr_1 \sin^q kr_1 .$$

Again we quote only those of the required integrals for which $\gamma \neq 0$.

$$I(\alpha,\beta,\gamma;\,1,0,0;\,0,1) = 2k\left\{\frac{2\alpha}{(\alpha^2-\gamma^2)^2}\left[\frac{1}{(\beta+\gamma)^2+k^2}-\frac{1}{(\alpha+\beta)^2+k^2}\right]\right.$$
$$\left.-\frac{1}{\alpha^2-\gamma^2}\left[\frac{2(\alpha+\beta)}{((\alpha+\beta)^2+k^2)^2}\right]\right\}. \tag{A.8}$$

$$I(\alpha,\beta,\gamma;\ 1,-1,1;\ 0,1) = (\alpha^2-\gamma^2)^{-2}\left\{\frac{16\alpha\gamma}{\alpha^2-\gamma^2}\ \tan\frac{\beta+\gamma}{k} + \frac{4k\alpha}{(\beta+\gamma)^2+k^2}\right\}$$

$$+\ \alpha \longleftrightarrow \gamma$$

$$I(\alpha,\beta,\gamma;\ 1,0,1,1,0) = \frac{\partial}{\partial k}\, I(\alpha,\beta,\gamma;\ 1,-1,1;\ 0,1) =$$

$$(\alpha^2-\gamma^2)^{-2}\left\{\frac{16\alpha\gamma}{\alpha^2-\gamma^2}\cdot\frac{\alpha+\beta}{(\alpha+\beta)^2+k^2}\right.$$

$$\left. + \frac{4\alpha}{(\beta+\gamma)^2+k^2} - \frac{8\alpha\ k^2}{\left[(\beta+\gamma)^2+k^2\right]^2}\right\}$$

$$+\ \alpha \longleftrightarrow \gamma$$

$$I(\alpha,\beta,\gamma;\ 1,0,0;\ 1,0) = \frac{4\alpha}{(\alpha^2-\gamma^2)^2}\left[\frac{\beta+\gamma}{(\beta+\gamma)^2+k^2} - \frac{\alpha+\beta}{(\alpha+\beta)^2+k^2}\right]$$

$$+ \frac{2}{\alpha^2-\gamma^2}\left[\frac{1}{(\alpha+\beta)^2+k^2} - \frac{2(\alpha+\beta)^2}{\left[(\alpha+\beta)^2+k^2\right]^2}\right]$$

$$I(\alpha,\beta,\gamma;\ 1,-1,1;\ 1,0) = \frac{8\alpha\gamma}{(\alpha^2-\gamma^2)^3}\cdot\log((\beta+\gamma)^2+k^2)$$

$$+ \frac{2\alpha}{(\alpha^2-\gamma^2)^2}\cdot\frac{2(\beta+\gamma)}{(\beta+\gamma)^2+k^2}$$

$$+\ \alpha \longleftrightarrow \gamma$$

$$I(\alpha,\beta,\gamma;\ 1,0,1;\ 0,1) = -\frac{\partial}{\partial k}\, I(\alpha,\beta,\gamma;\ 1,-1,1;\ 1,0) =$$

$$-\frac{16k\alpha\gamma}{(\alpha^2-\gamma^2)^3}\cdot\frac{1}{(\beta+\gamma)^2+k^2} + \frac{8k\alpha}{(\alpha^2-\gamma^2)^2}\cdot\frac{\beta+\gamma}{\left[(\beta+\gamma)^2+k^2\right]^2}$$

$$+\ \alpha \longleftrightarrow \gamma$$

Appendix B

We wish to consider here a point which arises in the generalization of the Kato formalism from the case of scattering by a center of force to the situation in which the scatterer is a compound system. Suppose we are dealing with a problem of the type discussed in Section 2. Then, according to the definition of the eigenfunctions $\phi_{n,\theta}$, (solutions of Eq. (2.9) with boundary conditions given by Eqs. (2.11)), one can easily construct a scattering problem for which the set of functions, $\left\{\phi_{n,\theta}\right\}$, is not complete. Thus, with $W_{12} = 0$ and $\rho = \rho(r_1)$, the eigenfunctions will each be of the form

$$\phi_{n,\theta} = \overline{R}(r_2)f(r_1) \quad .$$

The true solution to the real problem, $\overline{u}_\theta$, will also be of the product form. If one chooses a trial function, u_θ, which is not simply a product of $\overline{R}(r_2)$ with some function of r_1 then $w_\theta = u_\theta - \overline{u}_\theta$ will not be of the product form either and clearly can not be represented as an expansion in the $\phi_{n,\theta}$. Of course, this particular case represents no real difficulty since one can easily recognize the above-mentioned restriction on the form of $\phi_{n,\theta}$ and $\overline{u}_\theta$ and choose a trial function of the same form. It is possible, however, that the true solutions may satisfy some symmetry properties which are not easily recognizable. It would be desirable, therefore, to have a precise formulation of the conditions under which the set $\left\{\phi_{n,\theta}\right\}$ is complete. We shall not attempt to provide this here. However, some general conjectures along these lines may be made.

We observe first that the boundary conditions placed on the eigenfunctions in the one body problem, i.e.,

$$\text{(B.1)} \qquad \phi_{n,\theta}(r) \longrightarrow \text{const.}\ \sin(kr + \theta + n\pi)\ , \qquad \text{as } r_1 \longrightarrow \infty$$

$$\phi_{n,\theta}(0) = 0$$

are completely equivalent to the requirement that

$$\text{(B.2)} \qquad \int_0^\infty w_\theta \mathcal{L} \phi_{n,\theta}\, dr - \int_0^\infty \phi_{n,\theta} \mathcal{L}\, w_\theta\, dr = 0$$

where $\mathcal{L}$ is now the appropriate operator for the one body problem. Thus, a solution of the equation

$$\mathcal{L}\phi + \mu\rho\phi = 0$$

has the asymptotic form

$$\phi \longrightarrow C \sin(kr + \Theta) \qquad , \qquad \text{as } r \to \infty$$

where C is a non-vanishing constant (i.e., there are no bound states in the continuum). We may therefore write

$$\Delta \equiv \int_0^\infty w_\theta \mathcal{L} \phi\, dr - \int_0^\infty \phi \mathcal{L} w_\theta\, dr = (w_\theta \phi' - \phi w_\theta') \Big|_0^\infty$$

$$= \phi(0)\, w_\theta'(0) + kCD \sin(\theta - \Theta)$$

using the fact that $w_\theta(0) = 0$ and $w_\theta \longrightarrow D \sin(kr + \theta)$ as $r \longrightarrow \infty$, D being a non-vanishing constant. We note that $w_\theta'(0)$ is an arbitrary number depending on the particular choice of the trial function. Since $CD \neq 0$ we must therefore have $\phi(0) = 0$ and $\textcircled{H} = \theta$ in order that $\triangle$ vanish for an arbitrary function w_θ satisfying the boundary conditions. The equivalence of Eqs. (B.1) and (B.2) is then apparent. The generalizations of these equations in the many body problem are, however, not necessarily equivalent. To see this we need only return to the problem cited above, in which $W_{12} = 0$. For certain values of $\mu = \mu_n$ there exist solutions of the form

$$\phi_n = \bar{R}_{m,L}(r_2)\, F_{m,L}(r_1)\, P_L(p)$$

where $\bar{R}_{m,L}(r_2)$ represents an excited state for particle 2 and $F_{m,L}(r_1)$ is a bound state function (vanishing asymptotically). We note that this type of solution will satisfy the condition

$$\text{(B.3)} \qquad \int w_\theta \mathcal{L} \phi_n dt - \int \phi_n \mathcal{L} w_\theta \, dt = 0$$

since the surface terms vanish; however, they do not satisfy Eqs. (2.11). Further, the inclusion of these solutions in the set $\left\{\phi_{n,\theta}\right\}$ allows the possibility that the set will be complete even for trial functions not of the product form. It would therefore seem to be preferable, in general, to define the eigenfunctions ϕ_n as those solutions of Eq. (2.9) which satisfy Eq. (B.3). The question of whether this is sufficient to ensure completeness would still have to be studied, of course. Further, even assuming completeness, one would then have to investigate what effect, if any, this more general definition of the set $\left\{\phi_n\right\}$ had on the determination of the eigenvalues α_θ and β_θ. These considerations, however, do not alter the validity of our present results for the e^+H problem and it is hoped that in general they introduce no major obstacles to the practical application of the Kato method. This would be the case if, for example, the non-decaying eigenfunctions were complete for a well defined class of functions to which w_θ belongs.

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Research Reports and Publications

The following is a list of all research reports in the CX series.

CX-1	S. Borowitz	The Schwinger Variational Method for Three-Body Collisions Published - Borowitz and Friedman, Phys. Rev., 89, 441 (1953). Math. Rev., 14, 756 (1953).	April, '52
CX-2	H. Moses	A Self-Consistent Calculation of the Dissociation of Oxygen in the Upper Atmosphere Published - Phys. Rev., 87, 628 (1952).	April, '52
CX-3	K. Wildermuth	A Rigorous Solution of a Many-Body Problem Published - Acta Physica Austriaca, 7, 299 (1953).	Aug. '52
CX-4	B. Friedman and E. Gerjuoy	Scattering Problems in Nonrelativistic Quantum Mechanics; Part I-One Particle in One Dimension	Dec. '52
CX-5	H. Moses	Exchange Scattering in a Three-Body Problem Published - Phys. Rev., 91, 185 (1953). Math. Rev., 16, 1186 (1955).	Jan. '53
CX-6	H. Moses	Formulation of the Kohn-Hulthen Variational Principle in Terms of the Scattering Operator Formalism Published - Phys. Rev., 92, 817 (1953). Math. Rev., 15, 587 (1954).	April, '53
CX-7	H. Boyet and S. Borowitz	A Variational Calculation of the Elastic Scattering of Electrons by Hydrogen Atoms Published - Phys Rev., 93, 1225 (1954).	April, '53
CX-8	H. Moses	A Self-Consistent Calculation of the Dissociation of Oxygen in the Upper Atmosphere; Part II-Three-Body Recombinations Published - Phys. Rev., 91, 1408 (1953).	April, '53

CX-9	E. Bauer and Ta-You Wu	The Origin of the E-Layer of the Ionosphere Published - Phys. Rev., 92,1101 (1953).	July,'53
CX-10	J.B.Keller	Derivation of the Bohr-Sommerfeld Quantum Conditions from an Asymptotic Solution of the Schrödinger Equation Published - Annals of Physics; 4, 180 (1958).	July,'53
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CX-12	H. Moses	The Scattering Operator in Quantum Mechanics, Part I-The Properties of the Sdattering Operator from the Time-Dependent Schrödinger Equation (see CX-13) Math. Rev., 16, 1186 (1955).	Oct.,'53
CX-13	H. Moses	The Scattering Operator in Quantum Mechanics, Part II-The Scattering Operator Formalism and Other Formalisms Published-(CX-12 and CX-13, portion of the material)-Phys. Rev., 96, 519 (1954). Published-(CX-12 and CX-13,summary)-Il Nuovo Cimento, 1, 103 (1955). Math. Rev., 16,1187 (1955).	Dec.,'53
CX-14	E. Bauer and Ta-You Wu	The Cooling of a Gas by Radiation Published-Proc.Phys.Soc. of London, A 67, 741 (1954).	April,'54
CX-15	S. Schwebel	An Evaluation of Approximation Methods for Three-Body Scattering Problems Published - Phys.Rev., 103, 814 (1956). Math. Rev., 16, 1187 (1955). Math.Rev., 18, 701 (1957).	May,'54
CX-16	S. Borowitz	A Perturbation Calculation of the Elastic Scattering of Electrons by Hydrogen Atoms Published - Phys. Rev., 96, 1523 (1954).	June,'54

CX-17	E. Bauer	A Method of Calculating Cross Sections for Molecular Collisions Published - J. Chem. Phys., 23, 1087 (1955).	Oct., '54
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CX-20	H. Moses and I. Kay	The Determination of the Scattering Potential from the Spectral Measure Function, Part III-Calculation of the Scattering Potential from the Scattering Operator for the One-dimensional Schrödinger Equation Published - Il Nuovo Cimento, 3, 276 (1956). Math. Rev., 17, 489 (1956).	Sept., '55
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CX-27 (AD98722)	H. Moses	The Kohn-Hulthén Variational Procedure for the Scattering Operator and the Reactance Operator, Part II: Procedures Independent of the Normalization of the Trial Functions Published - Il Nuovo Cimento, 5, 144 (1957). Math. Rev., 18, 540 (1957).	July, '56
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CX-29 (AD117246)	H. Greenberg and S. Borowitz	A Variational Calculation of the Scattering Cross Section for Nearly Zero Energy Electrons by Hydrogen Atoms Published - Phys. Rev., 108, 716 (1957).	April, '57
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CX-36	H.E. Moses and S.F. Tuan	Potentials with Zero Scattering Phase	Nov., '58

List of Publications not Based on Reports

S. Borowitz and B. Friedman	Three-Body Scattering Problems Phys. Rev., 93, 251 (1954).
S. Borowitz	Electron-Ion Recombination at Low Pressures Trans. AIEE, 72, Part I: 430 (1953).
H. Moses	Calculation of the Scattering Potential from Reflection Coefficients Phys. Rev., 102, 559 (1956).
J. Keller, I. Kay and J. Shmoys	Determination of the Potential from Scattering Data Phys. Rev., 102, 557 (1956).
J. Keller	Bounds on Phase Shifts Published - Il Nuovo Cimento, 5, 1122 (1957).
H. Moses	Pathological Scattering in One Dimension Published - Il Nuovo Cimento, 5, 230 (1957).
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